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An a posteriori error estimate for finite element approximations of a singularly perturbed advection–diffusion problem

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Abstract

In this paper the author presents an a posteriori error estimator for approximations of the solution to an advection–diffusion equation with a non-constant, vector-valued diffusion coefficient ϵ in a conforming finite element space. Based on the complementary variational principle, we show that the error of an approximate solution in an associated energy norm is bounded by the sum of the weighted L^2-norms of solutions to a set of independent complementary variational problems, each defined on only one element of the partition. This error bound guarantees the over-estimation of the true error and does not depend unfavourably on ϵ as $\|\epsilon\|_\infty$ goes to zero. Although the original equation is a non-self-adjoint problem, the strong form of each local variational problem is always a Poisson equation with Neumann boundary conditions. The approximation of these local problems is then discussed and it is shown that, omitting a higher order term, the finite element solutions of these local complementary variational problems provide a computable upper error bound for the original finite element approximation in the energy norm. Numerical results, presented to validate the theoretical results, show that the computed error bounds are tight for a wide range of values of ϵ and always over-estimate the true errors.

Keywords: Finite element methods; A posteriori error estimates; Advection–diffusion equations; Singular perturbation

AMS classification: 65N30, 65N50

1. Introduction

Solutions of singularly perturbed advection–diffusion equations display sharp boundary or interior layers when the L^∞-norm $\|\epsilon\|_\infty$ of the singular perturbation parameter (diffusion coefficient) $\epsilon := (\varepsilon_1, \varepsilon_2)$ is much smaller than 1. Because of this difficulty, a singularly perturbed problem is often solved by a discretisation method in conjunction with an adaptive mesh refinement technique. The latter requires a reliable and efficient a posteriori error estimator. During the past decade two major types of a posteriori estimators have been developed: the element residual type of error estimates proposed in [4] and the postprocessing type proposed in [14]. Based on a complementary

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variational principle Kelly [7] proposed an error estimator of element residual type for the Galerkin finite element solution to a Laplace equation and showed that the real error in the energy norm is bounded by the sum of the energy norms of solutions to a set of local Neumann and Dirichlet problems, each defined on one element of the partitioning. In [2], Ainsworth and Oden present an error estimator for a self-adjoint problem based on a variational principle. Their work provides a theoretical analysis for the method in [7]. Other similar methods include [5]. There are some other methods for non-self-adjoint equations such as those of [6, 3].

Although there are many a posteriori error estimators for finite element approximations of conventional partial differential equations, very limited work has been done for the numerical solutions of singularly perturbed problems, especially in two and three dimensions. In [11], the authors extend the method in [2] to the incompressible Navier–Stokes equations. They showed that the error in a norm is bounded by the sum of the norms of the solutions to a set of singularly perturbed Poisson equations, each defined on an element of a mesh. Since the local problems are also singularly perturbed, the resulting estimated error bound (in an energy norm) may be large unless the local problems are solved on unpractically fine meshes. Some numerical results for a moderate value of ϵ can be found in [1].

In this paper we present an a posteriori error estimator for an important non-self-adjoint problem, i.e., a singularly perturbed advection-diffusion problem in 2 dimensions. Based on a complementary variational principle, we show that the error in an energy norm is bounded by the sum of the weighted L^2-norms of solutions to a set of independent variational problems, each defined on only one element of a given partition. Although the original problem is an advection–diffusion equation, each local problem is a Poisson equation with Neumann boundary conditions. Furthermore, the local problems do not depend explicitly on ϵ. Another notable feature of this method is that it uses only the interpolant in a conforming finite element space of a numerical solution obtained on a given mesh. So, using this method it is possible to construct a black box which takes a set of nodal approximations (obtained from an numerical method) and a set of mesh topology as its inputs and produces, as its output, an upper error bound for the interpolant of the nodal approximations in a conforming finite element space. As by-products, this paper provides a mathematical analysis for the heuristic results in [7] and an alternative analysis for the method discussed in [2], since the problems considered in both papers are special cases here. This paper is organised as follows.

The problem is stated in the next section. Based on a complementary variational principle, we show in Section 3 that the error of any approximation to the advection–diffusion equation in a conforming finite element space is bounded by a weighted L^2-norm of a solution to an associated complementary variational problem. In Section 4 we demonstrate that this complementary variational problem defined globally in the solution domain can be decomposed into a set of independent sub-problems, each defined on only one element of the partitioning. In Section 5 we discuss the approximation of these local problems in a finite element space larger than the original one. This provides an approximate bound which differs from the theoretical one by a higher order term than the error in the original finite element solution. In Section 6 we present some numerical results for the case that $\epsilon = (\varepsilon, \varepsilon)$ to validate the theoretical results. The numerical results show that the computed error bounds are tight for a wide range of values of ε and always over-estimate the true errors.

Although the method is described in 2 dimensions, it can be trivially extended to 3 dimensions.

2. The problem

Let us consider advection–diffusion problems of the form

$$Lu := -\nabla \cdot (\boldsymbol{\epsilon} \nabla u - \boldsymbol{b} u) + \mathrm{d} u = f \quad \text{in } \Omega \tag{2.1}$$

$$u = u_D \quad \text{on } \Gamma_D \tag{2.2}$$

$$G(u) := \boldsymbol{n} \cdot (\boldsymbol{\epsilon} \nabla u - \boldsymbol{b} u) = g \quad \text{on } \Gamma_N, \tag{2.3}$$

where $\boldsymbol{\epsilon} = (\varepsilon_1, \varepsilon_2)$ and $\boldsymbol{\epsilon q} := (\varepsilon_1 q_1, \varepsilon_1 q_2)$ for any vector-valued function $\boldsymbol{q} = (q_1, q_2)$. In the above $\Omega \subset \mathbb{R}^2$ is a bounded open set, $\Gamma_D \cap \Gamma_N = \emptyset$, $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \Gamma := \partial\Omega$, and $\boldsymbol{n}$ denotes the unit vector normal to $\partial\Omega$ in the outward direction. The boundary Γ is assumed to be piecewise smooth, without cusps.

In what follows $L^2(S)$ denotes the space of square-integrable functions on S, and $H^m(S)$ denotes the usual Sobolev space of L^2 functions with square integrable partial derivatives of up to order m, for any bounded open set $S \in \mathbb{R}^2$. If S is a curve the measure is arclength. The space $(L^2(S))^2$ is denoted by $\boldsymbol{L}^2(S)$. We use $\|\cdot\|_{0,S}$ to denote the norms on both $L^2(S)$ and $\boldsymbol{L}^2(S)$, and use $\|\cdot\|_{m,S}$ and $|\cdot|_{k,S}$ $(0 \leqslant k \leqslant m)$ to denote respectively the norm and the kth-order seminorm on $H^m(S)$. Let $(v, w)_S$ denote the integral $\int_S vw \, \mathrm{d}\Omega$ for any scalar or vector-valued functions v and w, so that $(\cdot,\cdot)_S$ is also the inner product on either $L^2(S)$ or $\boldsymbol{L}^2(S)$. When $S = \Omega$ we omit the subscript S. For any $k \geqslant 0$, we use $C^k(\Omega)$ (or $C^k(\overline{\Omega})$) to denote the set of functions ϕ such that ϕ and all its derivatives up to and including $\phi^{(k)}$ are continuous on Ω (or on $\overline{\Omega}$). We put $H^1_D(\Omega) = \{v \in H^1(\Omega)\colon v|_{\Gamma_D} = 0\}$. Other notation will be introduced when necessary.

Without loss of generality we assume that $u_D = 0$. The non-homogeneous case can be transformed into the homogeneous one by subtracting Lu_0 from both sides of (2.1), where u_0 is a known function satisfying the boundary condition (2.2). For the coefficient functions $\boldsymbol{\epsilon}, \boldsymbol{b}, d$ and the given data f, g we assume that $\boldsymbol{\epsilon} \in C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$, $\nabla \cdot \boldsymbol{b}, d, f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega_N)$, and that

$$\varepsilon_1, \varepsilon_1 \geqslant \sigma > 0 \quad \text{in } \Omega, \tag{2.4}$$

$$\boldsymbol{n} \cdot \boldsymbol{b} \leqslant 0 \quad \text{on } \Gamma_N \tag{2.5}$$

$$\nabla \cdot \boldsymbol{b} + 2d \geqslant 0 \quad \text{in } \Omega, \tag{2.6}$$

for some positive constants σ. For simplicity we assume that Γ is polygonal. We also assume that Γ_D has a positive measure (arclength). Obviously when $\varepsilon_1, \varepsilon_2 \ll 1$, the above problem is singularly perturbed. For all the theoretical results in the rest of the paper we do not assume that the diffusion coefficient $\boldsymbol{\epsilon}$ is a vector-valued constant. This, of course, contains the case that $\varepsilon_1 = \varepsilon_2 = \varepsilon$ for a constant ε as a special one.

As usual, the weak formulation corresponding to (2.1)–(2.3) is:

Problem 2.1. Find $u \in H^1_D(\Omega)$ such that for all $v \in H^1_D(\Omega)$

$$A(u, v) = (f, v) + (g, v)_{\Gamma_N}, \tag{2.7}$$

where $A(\cdot,\cdot)$ is a bilinear form defined by

$$A(v, w) = (\boldsymbol{\epsilon} \nabla v - \boldsymbol{b} v, \nabla w) + (dv, w) \quad \forall v, w \in H^1_D(\Omega). \tag{2.8}$$

Under the conditions (2.4)–(2.6) it can be shown that $\|\cdot\|_E := (A(\cdot,\cdot))^{1/2}$ is a norm on $H^1_D(\Omega)$ and thus Problem 2.1 is uniquely solvable. However, in most of our discussion below we shall use a different functional $\|\cdot\|$ defined by

$$\|v\|^2 := (\epsilon \nabla v - 2\boldsymbol{b}v, \nabla v) + 2(\mathrm{d}v, v) \quad \forall v \in H^1_D(\Omega). \tag{2.9}$$

The following theorem shows that $\|\cdot\|$ is a norm on $H^1_D(\Omega)$ and equivalent to the natural energy norm $\|\cdot\|_E$. The proof that $\|\cdot\|_E$ is a norm on $H^1_D(\Omega)$ is analogous to that for $\|\cdot\|$.

Lemma 2.1. *The functional $\|\cdot\|$ is a norm on $H^1_D(\Omega)$ satisfying*

$$\tfrac{1}{2}\|v\|^2 \leqslant \|v\|_E^2 \leqslant \|v\|^2 \quad \forall v \in H^1_D(\Omega). \tag{2.10}$$

Proof. To prove that $\|\cdot\|$ is a norm we need only to show that for any $v, w \in H^1_D(\Omega)$
1. $\|\lambda v\|_E = |\lambda| \|v\|_E$ for any $\lambda \in \mathbb{R}$,
2. $\|v\|_E \geqslant 0$ and $\|v\|_E = 0$ implies $v = 0$,
3. $\|v + w\|_E \leqslant \|v\|_E + \|w\|_E$.

From the definition (2.9) it is easy to show that 1 holds. Thus we consider 2. Integrating by parts we have

$$(\boldsymbol{b}v, \nabla v) = ((\boldsymbol{b}\cdot)\boldsymbol{n}v, v)_{\Gamma_N} - (\nabla \cdot \boldsymbol{b}v, v) - (\boldsymbol{b}v, \nabla v),$$

and so

$$(\boldsymbol{b}v, \nabla v) = \tfrac{1}{2}[(\boldsymbol{b} \cdot \boldsymbol{n}v, v)_{\Gamma_N} - (\nabla \cdot \boldsymbol{b}v, v)]. \tag{2.11}$$

Substituting this into (2.9) we obtain

$$\|v\|^2 = (\epsilon \nabla v, \nabla v) + ((\nabla \cdot \boldsymbol{b} + 2d)v, v) - (\boldsymbol{b} \cdot \boldsymbol{n}v, v)_{\Gamma_N}. \tag{2.12}$$

Thus, from (2.4)–(2.6) we see that $\|v\|^2 \geqslant 0$. Furthermore, $v = 0$ if $\|v\| = 0$, because $(\epsilon \nabla v, \nabla v)^{1/2}$ is by itself a norm on $H^1_D(\Omega)$.

The proof of the triangle inequality 3 is rather standard and thus omitted here.

We now show (2.10). From (2.8), (2.9) and (2.11) we have that for any $v \in H^1_D(\Omega)$,

$$\begin{aligned} \|v\|^2 &= A(v,v) - (\boldsymbol{b}v, \nabla v) + (dv, v) \\ &= A(v,v) + ((\tfrac{1}{2}\nabla \cdot \boldsymbol{b} + d)v, v) - \tfrac{1}{2}(\boldsymbol{b} \cdot \boldsymbol{n}v, v)_{\Gamma_N}. \end{aligned}$$

Thus the right-side inequality in (2.10) follows from this, (2.5) and (2.6).

Similarly we have

$$\begin{aligned} A(v,v) &= (\epsilon \nabla v - \boldsymbol{b}v, \nabla v) + (dv, v) \\ &= \tfrac{1}{2}[(\epsilon \nabla v - 2\boldsymbol{b}v, \nabla v) + 2(dv, v)] + \tfrac{1}{2}(\epsilon \nabla v, \nabla v) \\ &= \tfrac{1}{2}\|v\|^2 + \tfrac{1}{2}(\epsilon \nabla v, \nabla v) \\ &\geqslant \tfrac{1}{2}\|v\|^2. \qquad \square \end{aligned}$$

We now consider finite element solutions to Problem 2.1. Let $S_h^p \subset H_D^1(\Omega) \cap C^0(\overline{\Omega})$ be a piecewise infinitely smooth finite element space of approximation order p. The Galerkin problem corresponding to Problem 2.1 is:

Problem 2.2. Find $u_h \in S_h^p$ such that for all $v \in S_h^p$

$$A(u_h, v_h) = (f, v_h) + (g, v_h)_{\Gamma_N}.$$

Obviously Problem 2.2 also has a unique solution.

Let $e = u - U_h$ where u is the solution to Problem 2.1 and $U_h \in S_h^p$ is any approximation to u. One special choice is $U_h = u_h$, where u_h is the solution to Problem 2.2. Another interesting choice is that U_h is the S_h^p-interpolant of a solution from a numerical method other than Problem 2.2 (e.g. a finite difference method with a comparable approximation order as that of Problem 2.2). Subtracting $A(U_h, v)$ from both sides of (2.7) we have the following problem for e.

Problem 2.3. Find $e \in H_D^1(\Omega)$ such that for all $v \in H_D^1(\Omega)$

$$A(e, v) = (f, v) + (g, v)_{\Gamma_N} - A(U_h, v). \tag{2.13}$$

Theoretically the solution to this problem gives the exact error. However, in practice, solving (2.13) is equivalent to solving (2.7).

3. The upper bound for $\|e\|$

In the previous section we showed that the error e satisfies (2.13) which is equivalent to (2.7). In this section we show that $\|e\|$ can be bounded by a weighted $\boldsymbol{L}^2$-norm of a solution to the corresponding complementary variational problem. This method is based on the complementary variational principle (cf., e.g., [10]).

For any vector valued function $\boldsymbol{q} = (q_1, q_2)$ we let $\boldsymbol{\epsilon}^{-1}\boldsymbol{q} := (\varepsilon_1^{-1} q_1, \varepsilon_2^{-1} q_2)$. Introducing a new variable $\boldsymbol{p} = \boldsymbol{\epsilon}\nabla e$ we define the following primal mixed variational problem corresponding to Problem 2.3:

Problem 3.1. Find $[\boldsymbol{p}, e] \in \boldsymbol{L}^2(\Omega) \times H_D^1(\Omega)$ such that for all $[\boldsymbol{q}, v] \in \boldsymbol{L}^2(\Omega) \times H_D^1(\Omega)$

$$(\boldsymbol{\epsilon}^{-1}\boldsymbol{p}, \boldsymbol{q}) - (\nabla e, \boldsymbol{q}) = 0, \tag{3.1}$$

$$(\boldsymbol{p} - \boldsymbol{b}e, \nabla v) + (de, v) = R(f, g, U_h, v), \tag{3.2}$$

where

$$R(f, g, U_h, v) := (f, v) + (g, v)_{\Gamma_N} - A(U_h, v). \tag{3.3}$$

It is easy to see that Problem 2.3 is equivalent to Problem 3.1 in the sense that if e is a solution to Problem 2.3, then $[\boldsymbol{\epsilon}\nabla e, e]$ is a solution to Problem 3.1. Conversely, if $[\boldsymbol{p}, e]$ is a solution to

Problem 3.1, then e is a solution to Problem 2.3 and $\boldsymbol{p}=\epsilon\nabla e$. We now define a quadratic functional $\mathscr{G}$ on $\boldsymbol{L}^2(\Omega)\times H^1_D(\Omega)$ by

$$\mathscr{G}(\boldsymbol{q},v)=(\boldsymbol{q}-\boldsymbol{b}v,\nabla v)+\tfrac{1}{2}(dv,v)-\tfrac{1}{2}(\epsilon^{-1}\boldsymbol{q},\boldsymbol{q})-R(f,g,U_h,v). \tag{3.4}$$

Then we have the following lemma.

Lemma 3.1. *Let $[\boldsymbol{p},e]$ be a solution to Problem 3.1. Then for all $\boldsymbol{q}\in\boldsymbol{L}^2(\Omega)$ we have*

$$(\epsilon\nabla e,\nabla e)+(de,e)\leqslant-2\mathscr{G}(\boldsymbol{q},e). \tag{3.5}$$

Proof. Any $[\boldsymbol{q},v]\in\boldsymbol{L}^2(\Omega)\times H^1_D(\Omega)$ can be expressed as $[\boldsymbol{q},v]=[\boldsymbol{p}+\boldsymbol{\delta},e+\delta]$ with $(\boldsymbol{\delta},\delta)\in\boldsymbol{L}^2(\Omega)\times H^1_D(\Omega)$. Thus, from (3.4) we have, by direct computation,

$$\begin{aligned}\mathscr{G}(\boldsymbol{p}+\boldsymbol{\delta},e+\delta) &= [(\boldsymbol{p}-\boldsymbol{b}e,\nabla e)+\tfrac{1}{2}(de,e)-\tfrac{1}{2}(\epsilon^{-1}\boldsymbol{p},\boldsymbol{p})-R(f,g,U_h,e)]+I_1+I_2\\ &\quad+(\boldsymbol{\delta}-\boldsymbol{b}\delta,\nabla\delta)-(\boldsymbol{b}\delta,\nabla e)+\tfrac{1}{2}(d\delta,\delta)-\tfrac{1}{2}(\epsilon^{-1}\boldsymbol{\delta},\boldsymbol{\delta})\\ &=\mathscr{G}(\boldsymbol{p},e)+I_1+I_2\\ &\quad+(\boldsymbol{\delta}-\boldsymbol{b}\delta,\nabla\delta)-(\boldsymbol{b}\delta,\nabla e)+\tfrac{1}{2}(d\delta,\delta)-\tfrac{1}{2}(\epsilon^{-1}\boldsymbol{\delta},\boldsymbol{\delta}),\end{aligned} \tag{3.6}$$

where

$$I_1=(\nabla e,\boldsymbol{\delta})-(\epsilon^{-1}\boldsymbol{p},\boldsymbol{\delta}),$$

$$I_2=(\boldsymbol{p}-\boldsymbol{b}e,\nabla\delta)+(de,\delta)-R(f,g,U_h,\delta).$$

Since $[\boldsymbol{p},e]$ is a solution to Problem 3.1 and $[\boldsymbol{\delta},\delta]\in\boldsymbol{L}^2(\Omega)\times H^1_D(\Omega)$ we have from (3.1) and (3.2) that $I_1=I_2=0$. Thus (3.6) reduces to

$$\mathscr{G}(\boldsymbol{p}+\boldsymbol{\delta},e+\delta)=\mathscr{G}(\boldsymbol{p},e)+(\boldsymbol{\delta}-\boldsymbol{b}\delta,\nabla\delta)-(\boldsymbol{b}\delta,\nabla e)+\tfrac{1}{2}(d\delta,\delta)-\tfrac{1}{2}(\epsilon^{-1}\boldsymbol{\delta},\boldsymbol{\delta}).$$

Setting $\delta=0$ in the above equality and using (2.4) we obtain

$$\mathscr{G}(\boldsymbol{q},e)=\mathscr{G}(\boldsymbol{p},e)-\tfrac{1}{2}(\epsilon^{-1}\boldsymbol{\delta},\boldsymbol{\delta})\leqslant\mathscr{G}(\boldsymbol{p},e), \tag{3.7}$$

since $\boldsymbol{p}+\boldsymbol{\delta}=\boldsymbol{q}$. Now, from (3.4) we have

$$\begin{aligned}\mathscr{G}(\boldsymbol{p},e) &= (\boldsymbol{p}-\boldsymbol{b}e,\nabla e)+\tfrac{1}{2}(de,e)-\tfrac{1}{2}(\epsilon^{-1}\boldsymbol{p},\boldsymbol{p})-R(f,g,U_h,e)\\ &= -\tfrac{1}{2}[(\epsilon^{-1}\boldsymbol{p},\boldsymbol{p})+(de,e)]\\ &= -\tfrac{1}{2}[(\epsilon\nabla e,\nabla e)+(de,e)],\end{aligned}$$

because $\boldsymbol{p}=\epsilon\nabla e$. In the above we used (3.2) since $e\in H^1_D(\Omega)$. Finally, combining this equality with (3.7) we obtain (3.5). □

Before stating the main theorems of this section we first define the following (complementary) variational problem.

Problem 3.2. Find $\boldsymbol{q} \in \boldsymbol{L}^2(\Omega)$ such that for all $v \in H_D^1(\Omega)$

$$(\boldsymbol{q}, \nabla v) = R(f, g, U_h, v), \tag{3.8}$$

where R is defined in (3.3).

The error bound for $\|e\|$ is established in the following theorem.

Theorem 3.1. *Let e be the solution to Problem 2.3 and $\boldsymbol{q}$ be a solution to Problem 3.2. Then we have*

$$\|e\| \leqslant (\boldsymbol{\epsilon}^{-1}\boldsymbol{q}, \boldsymbol{q})^{1/2} \tag{3.9}$$

Proof. Combining (3.5) and (3.4) we get

$$(\boldsymbol{\epsilon}\nabla e, \nabla e) + (de, e) \leqslant (\boldsymbol{\epsilon}^{-1}\boldsymbol{q}, \boldsymbol{q}) - (de, e) + 2(\boldsymbol{b}e, \nabla e) - 2[(\boldsymbol{q}, \nabla e) - R(f, g, U_h, e)],$$

because $\boldsymbol{q} \in \boldsymbol{L}^2(\Omega)$. From this, (2.5), (2.6) and (2.9) we have

$$\|e\|^2 \leqslant (\boldsymbol{\epsilon}^{-1}\boldsymbol{q}, \boldsymbol{q}) - 2[(\boldsymbol{q}, \nabla e) - R(f, g, U_h, e)]. \tag{3.10}$$

Since $e \in H_D^1(\Omega)$ and $\boldsymbol{q}$ satisfies (3.8), the last term in the above vanishes. So, taking square root on both side of the above we obtain (3.9). □

Theorem 3.1 shows that the weighted $\boldsymbol{L}^2$-norm of any solution to Problem 3.2 (the complementary variational problem) over-estimates $\|e\|$. From the right-hand side of (3.9) we see that the error bound depends on $\boldsymbol{\epsilon}^{1/2}$. Thus, a difficulty becomes apparent when either ε_1 or $\varepsilon_2 \ll 1$, i.e. the error bound will become unpractically large when $\varepsilon_1 \ll 1$ or $\varepsilon_2 \ll 1$. An improved error bound for the case that one or both of ε_1 and ε_2 is small is established in the following theorem.

Theorem 3.2. *Let e be the solution to Problem 2.3 and $\boldsymbol{q}$ a solution to Problem 3.2. Then, for each ε, when h is sufficiently small, there exists a positive constant M such that*

$$\|e\| \leqslant [(\boldsymbol{\epsilon}\boldsymbol{q}, \boldsymbol{q}) + M\|(\boldsymbol{I} - \boldsymbol{\epsilon})\boldsymbol{q}\|_0]^{1/2} \tag{3.11}$$

with $\boldsymbol{I} = (1, 1)$.

Proof. Note that $\boldsymbol{q}$ in (3.10) can be an arbitrary element in $\boldsymbol{L}^2(\Omega)$. On the substitution of $\boldsymbol{q}$ in (3.10) by $\boldsymbol{\epsilon q}$ we have that for all $\boldsymbol{q} \in \boldsymbol{L}^2(\Omega)$,

$$\|e\|^2 \leqslant (\boldsymbol{\epsilon}\boldsymbol{q}, \boldsymbol{q}) - 2[(\boldsymbol{\epsilon}\boldsymbol{q}, \nabla e) - R(f, g, U_h, e)]. \tag{3.12}$$

Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\|e\|^2 &\leqslant (\boldsymbol{\epsilon}\boldsymbol{q}, \boldsymbol{q}) - 2[(\boldsymbol{q}, \nabla e) - R(f, g, U_h, e)] + 2((\boldsymbol{I} - \boldsymbol{\epsilon})\boldsymbol{q}, \nabla e) \\ &\leqslant (\boldsymbol{\epsilon}\boldsymbol{q}, \boldsymbol{q}) + 2\|(\boldsymbol{I} - \boldsymbol{\epsilon})\boldsymbol{q}\|_0 \|\nabla e\|_0 - 2[(\boldsymbol{q}, \nabla e) - R(f, g, U_h, e)] \\ &= (\boldsymbol{\epsilon}\boldsymbol{q}, \boldsymbol{q}) + 2\|(\boldsymbol{I} - \boldsymbol{\epsilon})\boldsymbol{q}\|_0 \|\nabla e\|_0,\end{aligned}$$

if $\boldsymbol{q}$ is a solution to Problem 3.2. Since for any given ϵ, $\|\nabla e\|_0 \to 0$ as $h \to 0^+$. Thus, when h is sufficiently small, we have $2\|\nabla e\|_0 \leqslant M$ for a constant M. Combining this with the above inequality we obtain (3.11).

We comment that in general, the constant M depends on h in a favorable way and on ε in an unfavorable way. The dependence of M on h and ε depends on the discretisation method used for obtaining U_h. In the case that the discretisation method converges uniformly in ε, or that h is dependent of ε, M can be made to be independent of both h and ε. The upper bound in (3.11) is not sharp unless $\epsilon = I$. This is because of the application of the Cauchy–Schwarz inequality. So, in practice, M in (3.11) may be used as a parameter to control the tightness of the error bound. As will be seen later, for a wide range of values of ϵ, the choice of $M = 1$ gives tight computed error bounds for the discretisation scheme proposed in [8].

From (3.12) we can see that if choose $\boldsymbol{q}$ such that

$$(\epsilon \boldsymbol{q}, \nabla v) = R(f, g, U_h, v) \quad \forall v \in H_D^1(\Omega), \tag{3.13}$$

then we have

$$\|e\|^2 \leqslant (\epsilon \boldsymbol{q}, \boldsymbol{q}). \tag{3.14}$$

From their right sides we may think that (3.14) should give tighter bounds than (3.9). However, this is not the case, because now (3.13) is singularly perturbed, while (3.8) is not. In fact it is easy to see that (3.13)–(3.14) is equivalent to (3.8)–(3.9). The estimate (3.14) along with (3.13) is similar to the result in [11]. This, in practice, may result in wide error bounds in the energy norm $\|\cdot\|$ when $\|\epsilon\|_\infty \ll 1$.

Finally we comment that both Theorems 3.1 and 3.2 hold if the space $H_D^1(\Omega)$ in Problem 3.2 is replaced by $H^1(\Omega)$. It is because $H_D^1(\Omega) \subset H^1(\Omega)$. This will turn to be advantageous, as will be seen later.

4. Localisation of the complementary variational problem

In the previous section we showed that $\|e\|$ is bounded by the weighted L^2-norm of any solution to Problem 3.2. In this section we demonstrate that solutions to a set of localised problems form a solution to Problem 3.2. We first introduce some notation.

Let Ω_i and Γ_i $(i = 1, 2, \ldots, N)$ denote respectively the elements and their boundaries on which S_h^p is constructed, and E_j $(j = 1, 2, \ldots, M)$ the edges of the mesh. We use I to denote the set of all internal edges of the mesh, and O_D and O_N the sets of edges that make up the boundaries Γ_D and Γ_N, respectively. To each internal edge $E_j \in I$ we arbitrarily associate a unit normal $\boldsymbol{n}$, whereas when referring to boundary edges $E_j \in O_D \cup O_N$ or to the closed contours Γ or Γ_i $(i = 1, 2, \ldots, N)$, we consistently use $\boldsymbol{n}$ to be the outward normal. (The direction of the normal to an internal edge can therefore change when the edge is considered not in isolation but as a component of a closed contour Γ_i, but this causes no difficulty.)

Integrating $A(U_h, v)$ by parts we have from (3.8)

$$(\boldsymbol{q}, \nabla v) = (f, v) + (g, v)_{\Gamma_N} - \sum_{i=1}^{N} [(G(U_h), v)_{\Gamma_i} + (LU_h, v)_{\Omega_i}]$$

$$= \sum_{i=1}^{N} (r, v)_{\Omega_i} + (g, v)_{\Gamma_N} - \sum_{i=1}^{N} (G(U_h), v)_{\Gamma_i}, \tag{4.1}$$

where $r := f - LU_h$ and G is the trace operator defined in (2.3). We define the jump in the normal derivative of U_h on edges by

$$\delta\left(\frac{\partial U_h}{\partial \boldsymbol{n}}\right)_{E_j} = \begin{cases} \frac{\partial U_h}{\partial \boldsymbol{n}}\Big|_{E_j - 0} - \frac{\partial U_h}{\partial \boldsymbol{n}}\Big|_{E_j + 0} & \text{if } E_j \in I, \\ \frac{\partial U_h}{\partial \boldsymbol{n}} - g & \text{if } E_j \in O_N. \end{cases}$$

Thus, summing the last term over edges we have from (4.1)

$$(\boldsymbol{q}, \nabla v) = \sum_{i=1}^{N} (r, v)_{\Omega_i} - \sum_{j=1}^{M} \left(\delta\left(\frac{\partial U_h}{\partial \boldsymbol{n}}\right), v\right)_{E_j}. \tag{4.2}$$

We comment that the jump on Γ_D can be defined arbitrarily because $v|_{\Gamma_D} = 0$. Now, for each $i = 1, 2, \ldots, N$, we define a function J_i on Γ_i such that for any neighbouring element Ω_k of Ω_i

$$-\delta\left(\frac{\partial U_h}{\partial \boldsymbol{n}}\right)_{\Gamma_i \cap \Gamma_k} = \begin{cases} J_i + J_k & \text{if } \Gamma_i \cap \Gamma_k \in I, \\ J_i & \text{if } \Gamma_i \cap \Gamma_N \in O_N. \end{cases} \tag{4.3}$$

In fact, J_i and J_k form a splitting of $-\delta(\frac{\partial U_h}{\partial \boldsymbol{n}})$ on the edge $\Gamma_i \cap \Gamma_k$. Using J_i, the last term in (4.2) can be rewritten as

$$-\sum_{j=1}^{M} \left(\delta\left(\frac{\partial U_h}{\partial \boldsymbol{n}}\right), v\right)_{E_j} = \sum_{i=1}^{N} (J_i, v)_{\Gamma_i}.$$

Substituting this into (4.2) we have

$$(\boldsymbol{q}, \nabla v) = \sum_{i=1}^{N} [(r, v)_{\Omega_i} + (J_i, v)_{\Gamma_i}]$$

or

$$\sum_{i=1}^{N} [(\boldsymbol{q}, \nabla v)_{\Omega_i} - (r, v)_{\Omega_i} - (J_i, v)_{\Gamma_i}] = 0 \tag{4.4}$$

for all $v \in H_D^1(\Omega)$. For any $i = 1, 2, \ldots, N$, let $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i): v|_{\Gamma_i \cap \Gamma_D} = 0 \text{ if } \Gamma_i \cap \Gamma_D \neq \emptyset\}$. The above equation motivates us to seek $\boldsymbol{q} \in \boldsymbol{L}^2(\Omega)$ such that $\boldsymbol{q}_i := \boldsymbol{q}|_{\Omega_i}$ satisfies the following problem.

Problem 4.1. Find $\boldsymbol{q}_i \in \boldsymbol{L}^2(\Omega_i)$ such that for all $v \in H_D^1(\Omega_i)$

$$(\boldsymbol{q}_i, \nabla v)_{\Omega_i} = (r, v)_{\Omega_i} + (J_i, v)_{\Gamma_i}, \quad i = 1, 2, \ldots, N. \tag{4.5}$$

Since $v \in H_D^1(\Omega)$ implies that $v|_{\Omega_i} \in H_D^1(\Omega_i)$, we see that any solution to Problem 4.1 is a solution to Problem 3.2. Furthermore, if $\Gamma_i \cap \Gamma_D = \emptyset$, (4.5) is a Neumann problem, and it has a solution if the compatibility condition

$$\int_{\Gamma_i} J_i \, \mathrm{d}\Gamma + \int_{\Omega_i} r \, \mathrm{d}\Omega = 0 \tag{4.6}$$

is satisfied. Because any $\boldsymbol{q}$ such that $\boldsymbol{q}_i$ satisfies (4.5) is a solution to Problem 3.2, and thus satisfies (3.9) and (3.11), we seek a particular $\boldsymbol{q}$ which is locally irrotational, i.e. there exists a scalar function $\phi_i \in H_D^1(\Omega_i)$ such that $\boldsymbol{q}_i = \nabla \phi_i$ for all $i = 1, 2, \ldots, N$. Thus, for this special case, Problem 4.1 can be restated as

Problem 4.2. Find $\phi_i \in H_D^1(\Omega_i)$ such that for all $v \in H_D^1(\Omega_i)$

$$B(\phi_i, v) = (r, v)_{\Omega_i} + (J_i, v)_{\Gamma_i}, \quad i = 1, 2, \ldots, N, \tag{4.7}$$

with $B(\cdot, \cdot)$ a bilinear form defined by

$$B(v, w) = (\nabla v, \nabla w)_{\Omega_i} \quad \forall v, w \in H_D^1(\Omega_i). \tag{4.8}$$

For any i, if $\Gamma_i \cap \Gamma_D = \emptyset$, then (4.7) is a Neumann problem which has a solution provided that (4.6) is satisfied. If part of Γ_i is on Γ_D, Problem 4.2 has a unique solution ϕ_i satisfying $\phi_i|_{\Gamma_i \cap \Gamma_D} = 0$. In both cases $\nabla \phi_i$ is uniquely determined although ϕ_i itself may not be unique. The following theorem shows that the gradient of a solution to Problem 4.2 minimises the $\boldsymbol{L}^2$-norm of all solutions to Problem 4.1.

Theorem 4.1. *Let $\boldsymbol{q}_i$ and ϕ_i be solutions to Problem* 4.1 *and Problem* 4.2, *respectively. Then we have*

$$\|\nabla \phi_i\|_{0,\Omega_i} \leqslant \|\boldsymbol{q}_i\|_{0,\Omega_i}, \quad i = 1, 2, \ldots, N. \tag{4.9}$$

Proof. From (4.7) and (4.5) we have

$$\|\nabla \phi_i\|_{0,\Omega_i}^2 = (r, v)_{\Omega_i} + (J_i, v)_{\Gamma_i} = (\boldsymbol{q}_i, \nabla \phi_i)_{0,\Omega_i} \leqslant \|\boldsymbol{q}_i\|_{0,\Omega_i} \|\nabla \phi_i\|_{0,\Omega_i}.$$

Thus, (4.9) follows from this. □

Therefore, instead of solving Problem 4.1 we can also solve Problem 4.2 for ϕ_i, and $\nabla \phi_i$ will give a tighter upper bound for $\|e\|$ than any other solution to Problem 4.1.

We may also define a problem similar to Problem 4.2 such that all local problems have Neumann boundary conditions. Since the jump on Γ_D is arbitrary, we choose $\delta(\partial U_h / \partial \boldsymbol{n})_{E_i} = 0$ if $E_i \in O_D$ and the splitting of the jump on edges in O_D to be the same as the one for edges in O_N given in (4.3). Thus we define

Problem 4.3. Find $\phi_i \in H^1(\Omega_i)$ such that for all $v \in H^1(\Omega_i)$

$$B(\phi_i, v) = (r, v)_{\Omega_i} + (J_i, v)_{\Gamma_i}, \quad i = 1, 2, \ldots, N, \tag{4.10}$$

with B the bilinear form defined in (4.8).

For each $i=1,2,\ldots,N$, Problem 4.3 is solvable provided that (4.6) is satisfied. Because $H_D^1(\Omega_i)\subset H^1(\Omega_i)$, it is easy to see that if ϕ_i^* is a solution to Problem 4.3, then ϕ_i^* satisfies (4.7) and $\boldsymbol{q}_i^* := \nabla\phi_i^*$ satisfies (4.5) for all $v\in H^1(\Omega_i)$. Let $\boldsymbol{q}^*$ be the function such that $\boldsymbol{q}^*|_{\Omega_i}=\boldsymbol{q}_i^*$. Then $\boldsymbol{q}^*$ is a solution to Problem 3.2 with $H_D^1(\Omega)$ replaced by $H^1(\Omega)$. As commented at the end of the previous section, this $\boldsymbol{q}^*$ also provides an upper error bound for $\|e\|$ through Theorem 3.1 or 3.2. Now it is apparent that Theorem 4.1 still holds for ϕ_i^* and solutions to Problem 4.1 with $H_D^1(\Omega_i)$ replaced by $H^1(\Omega_i)$. If we let ϕ_i be the solution to Problem 4.2, then we expect that ϕ_i^* provides a tighter upper bound for $\|e\|$ than ϕ_i. This is because, by Theorem 4.1, ϕ_i^* is the minimiser in $H^1(\Omega_i)$ while ϕ_i is the minimiser in the subspace $H_D^1(\Omega_i)$ of $H^1(\Omega_i)$.

5. Approximation of the upper error bound

In the previous sections we showed that the error $\|e\|$ can be bounded by the weighted $\boldsymbol{L}^2$-norm of the gradient of a solution to Problem 4.2 or 4.3. However these problems can not be solved exactly except for some special cases. Therefore we look for approximations to Problem 4.2 or Problem 4.3. This in turn gives approximate upper error bounds for $\|e\|$. Both of the local problems are solvable when the boundary condition J_i is properly determined. Since a set $\{J_i\}_1^N$ satisfying (4.3) and (4.6) is far from unique, computationally we seek the solution with the minimum L^2-norm. For detail of this discussion we refer to [13]. We now concentrate on the approximation of Problem 4.2. All the following results hold for the approximation of Problem 4.3.

For $i=1,2,\ldots,N$, let $S_{i,h_*}^{p+q}\subset H_D^1(\Omega_i)\cap C^0(\overline{\Omega}_i)$ be a piecewise infinitely smooth finite element space with $h_*\leqslant h$ and $q\geqslant 0$ an integer. Obviously, S_{i,h_*}^{p+q} can be an h-, p- or h-p-version refinement of the space $S_{i,h}^p := S_h^p|_{\overline{\Omega}_i}$. Using this space we define the following Galerkin problem.

Problem 5.1. Find $\phi_{i,h_*}\in S_{i,h_*}^{p+q}$ such that for all $v_{h_*}\in S_{i,h_*}^{p+q}$

$$B(\phi_{i,h_*},v_{h_*})=(r,v_{h_*})_{\Omega_i}+(J_i,v_{h_*})_{\Gamma_i},\quad i=1,2,\ldots,N, \tag{5.1}$$

where $B(\cdot,\cdot)$ is the bilinear form defined by (4.8).

Using the standard argument it is easy to show that

$$\left(\sum_{i=1}^N \|\nabla(\phi_i-\phi_{i,h_*})\|_{0,\Omega_i}^2\right)^{1/2} \leqslant Ch_*^{p+q-1}\left(\sum_{i=1}^N |\phi_i|_{p+q,\Omega_i}^2\right)^{1/2}, \tag{5.2}$$

where $C>0$ is a constant, independent of h_* and ϕ_i. This estimate also holds if ϕ_{i,h_*} is replaced by the S_{i,h_*}^{p+q}-interpolant of ϕ_i. Now, if we use the solution ϕ_{i,h_*} of Problem 5.1 to evaluate the right side of (3.9) or (3.11), then some new errors are introduced because ϕ_{i,h_*} does not solve Problem 3.2 exactly. The following theorem shows that the error due to the approximation of Problem 4.2 by Problem 5.1 is of an higher order, and thus gives computable error bounds for $\|e\|$ corresponding to Theorem 3.1 and Theorem 3.2, respectively.

Theorem 5.1. *Let e be the solution to Problem* 2.3, *and let ϕ_i and ϕ_{i,h_*} be the solutions to Problem* 4.2 *and Problem* 5.1, *respectively. Then we have*

$$\|e\| \leqslant K^{1/2}(\phi_{i,h_*}) + Ch_*^{p+q-1}\left(\sum_{i=1}^{N} |\phi_i|^2_{p+q,\Omega_i} \sum_{i=1}^{N} |e|^2_{p+q,\Omega_i}\right)^{1/4}, \tag{5.3}$$

where $C>0$ is a constant, independent of h_*, ϵ, ϕ_i and e, and

$$K(\phi_{i,h_*}) = \sum_{i=1}^{N} (\epsilon^{-1}\nabla\phi_{i,h_*}, \nabla\phi_{i,h_*})$$

or

$$K(\phi_{i,h_*}) = \sum_{i=1}^{N} [(\epsilon\nabla\phi_{i,h_*}, \nabla\phi_{i,h_*}) + M\|(\boldsymbol{I}-\epsilon)\nabla\phi_{i,h_*}\|_0].$$

Here M and $\boldsymbol{I}$ are the same as those in Theorem 4.2.

Proof. Let C be a generic positive constant, independent of h_*, ϕ_i and e, and $\boldsymbol{q}_{h_*} \in \boldsymbol{L}^2(\Omega)$ be a vector-valued function such that

$$\boldsymbol{q}_{h_*}|_{\Omega_i} = \nabla\phi_{i,h_*}, \quad i = 1,2,\ldots,N.$$

Since $\boldsymbol{q}_{h_*} \in \boldsymbol{L}^2(\Omega)$ we have that it satisfies (3.10) and (3.12). For both cases we have

$$\|e\|^2 \leqslant K(\phi_{i,h_*}) - 2[(\boldsymbol{q}_{h_*}, \nabla e) - R(f,g,U_h,e)]. \tag{5.4}$$

Let e_i^I be the S_{i,h_*}^{p+q}-interpolant of $e|_{\Omega_i}$. Using the same technique for the deduction of (4.4) we have

$$\begin{aligned}
|(\boldsymbol{q}_{h_*}, \nabla e) - R(f,g,U_h,e)| &= \left|\sum_{i=1}^{N} [B(\phi_{i,h_*}, e) - (r,e)_{\Omega_i} - (J_i,e)_{\Gamma_i}]\right| \\
&= \left|\sum_{i=1}^{N} [B(\phi_{i,h_*}, e-e_i^I) - (r,e-e_i^I)_{\Omega_i} - (J_i,e-e_i^I)_{\Gamma_i}]\right| \\
&= \left|\sum_{i=1}^{N} B(\phi_{i,h_*}-\phi_i, e-e_i^I)\right| \\
&\leqslant \sum_{i=1}^{N} \|\nabla(\phi_{i,h_*}-\phi_i)\|_{0,\Omega_i}\|\nabla(e-e_i^I))\|_{0,\Omega_i} \\
&\leqslant \left(\sum_{i=1}^{N} \|\nabla(\phi_{i,h_*}-\phi_i)\|^2_{0,\Omega_i}\right)^{1/2}\left(\sum_{i=1}^{N} \|\nabla(e-e_i^I)\|^2_{0,\Omega_i}\right)^{1/2} \\
&\leqslant Ch_*^{2(p+q-1)}\left(\sum_{i=1}^{N} |\phi_i|^2_{p+q,\Omega_i} \sum_{i=1}^{N} |e|^2_{p+q,\Omega_i}\right).
\end{aligned}$$

In the above we used (5.1), (4.7), (5.2) and the Cauchy–Schwarz inequality. Combining the above with (5.4) we obtain

$$\|e\|^2 \leqslant K(\phi_{i,h_*}) + Ch_*^{2(p+q-1)} \left(\sum_{i=1}^{N} |\phi_i|^2_{p+q,\Omega_i} \sum_{i=1}^{N} |e|^2_{p+q,\Omega_i} \right)^{1/2}$$

$$\leqslant \left[K^{1/2}(\phi_{i,h_*}) + Ch_*^{p+q-1} \left(\sum_{i=1}^{N} |\phi_i|^2_{p+q,\Omega_i} \sum_{i=1}^{N} |e|^2_{p+q,\Omega_i} \right)^{1/4} \right]^2.$$

Taking square root on both sides of the above we obtain (5.3). □

Theorem 5.1 shows that the computed bound $K(\phi_{i,h_*})$ for $\|e\|$ is only an approximation to the upper error bound defined by any exact solution to Problem 4.2. The error of this computable upper bound is of order h_*^{p+q-1}. Since $\|e\|$ itself is of order h^{p-1}, it is essential to choose either $h_* < h$ or $q > 0$, or both so that the last term on the right-hand side of (5.3) is of a higher order than that of $\|e\|$.

Finally we comment that there are some other methods for the approximation of the local problems. For example, the strong form corresponding to (4.5) can be approximated by the method proposed in [7] on a rectangular mesh. For details we refer to [7].

6. Numerical results

To verify the theoretical results established in the previous section some numerical experiments were carried out. All computations were performed in double precision on a Unix workstation.

The test problem is chosen to be the following.

Test: $-\nabla \cdot (\varepsilon \nabla u - \boldsymbol{b} u) = f \quad \text{in } \Omega = (0,1)^2,$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega,$$

with $\boldsymbol{b} = (1,1)$ and the exact solution

$$u = xy(1 - \mathrm{e}^{(x-1)/\varepsilon})(1 - \mathrm{e}^{(y-1)/\varepsilon}).$$

The right-hand side function is

$$f = x(1 - \mathrm{e}^{(x-1)/\varepsilon})(1 + \mathrm{e}^{(y-1)/\varepsilon}) + y(1 - \mathrm{e}^{(y-1)/\varepsilon})(1 + \mathrm{e}^{(x-1)/\varepsilon}).$$

This problem has two boundary layers along $x = 1$ and $y = 1$. From (2.12) it is easy to see that if u is a solution to the test problem, then the $\|u\|$ is identical to the energy norm $\|u\|_E$, and both of these are identical to the norm $\sqrt{\varepsilon}\|\nabla u\|_0$ on $H^1_D(\Omega)$. This is because $\nabla \cdot \boldsymbol{b} = d = 0$ and $\Gamma_N = \emptyset$. The solution domain Ω is covered by a 20×20 uniform square mesh ($h = 1/19$). The space S_h^p is chosen to be the conforming finite element space constructed using the conventional piecewise bilinear basis functions on this square mesh. Now, the test problem is first solved by the non-conforming finite element method proposed by [8] on the mesh, and then the S_h^p-interpolant of the solution from the non-conforming finite element method is used as an approximate solution U_h to the test problem in S_h^p. The error e for the numerical solution u_h from the non-conforming finite element method is given by (cf. [8])

$$h^{1/2}\|e\|_{1,h} \leqslant C\,|\boldsymbol{f}|_{1,h}h^{1/2},$$

where C is a positive constant which equals the minimum value of the projections of $\boldsymbol{b}$ on the edges of the mesh, $\boldsymbol{f}=\varepsilon\nabla u - \boldsymbol{b}u$ is the flux, $\|e\|_{1,h}$ is a discrete analogue of the H^1-norm $\|\nabla e\|_0$ and $|\cdot|_{1,h}$ is a discrete first order seminorm. (We may regard $h^{1/2}\|\cdot\|_{1,h}$ as an ε-independent energy norm on $H_D^1(\Omega)$.) This gives $\|e\|_{1,h} \leqslant C|\boldsymbol{f}|_{1,h}$. Although theoretically this error bound still depends on ε through the term $|\boldsymbol{f}|_{1,h}$, the numerical results in [9] demonstrate that the $h^{1/2}$-order convergence is independent of ε, at least when $\varepsilon \ll h$. Also, because $\boldsymbol{b}=(1,1)$ and the edges are all parallel to one of the coordinate axes, we have that $C=1$ in the above estimate. Thus we choose $M=1$ in (3.11) for all the relevant results below.

To solve the local complementary problem Problem 4.2 or 4.3. We first find $J_i (i=1,2,\ldots,N)$ such that the compatibility condition (4.6) is satisfied for all $i=1,2,\ldots,N$. Since the number of edges is greater than the number of elements, there are infinite sets of J_i satisfying (4.5). So, computationally we look for the one with minimum Euclidean norm. This is achieved by applying the LSQR algorithm proposed in [12] to this case. For details of this discussion we refer to [13]. After the determination of J_i, we solve Problem 5.1 for an approximation to Problem 4.2 as follows: Each element is divided into 4×4 sub-elements and the finite element space S_{i,h_*}^{p+q} in Problem 5.1 is chosen to be the span of the conventional 4-node bilinear elements on this refined mesh. In this case we have that $p=2, q=0$ and $h_*=h/4$. All the integrals in the finite element method are approximated by the 9-point Gauss quadrature rule in each element. The norm $\|u-U_h\|$ has to be evaluated with care when ε is small. This is because when $\varepsilon \ll 1$, the Gauss quadrature points in an element containing part of a boundary layer may all be outside of the layer. In this case, the numerical value of $\|u-U_h\|$ may be much smaller than the exact one. We avoid this by dividing the element into two (or three for the element containing the corner (1,1)) sub-elements such that one (or two) sub-element has a width (or/and height) of 4ε (recall that the widths of boundary layers of the Test are of O(ε) order). Table 1 is a list of the values of the exact norm $\|u-U_h\|$, the computed upper bounds for $\|u-U_h\|$ from (3.11) using both Problems 4.2 and 4.3 and the effectivity index γ (ratio of the estimated to the true error) for different values of ε. From this table we see that the computed error bounds are tight except for the case that $\varepsilon/h \approx$ O(1), and always over-estimate the true error, as proved in the previous sections. It is also seen that the solution to Problem 4.3 gives tighter upper error bound than that of Problem 4.2, as commented before.

Table 2 is a list of the exact energy norms, the error bounds and the effectivity indices for different values of ε computed using (3.9) and Problem 4.3. From this table we see that (3.9) gives unpractically wide bounds when ε is small.

7. Conclusions

In this paper we presented an a posteriori error estimator for approximations of a singularly perturbed advection–diffusion equation in a conforming finite element space. Based on a complementary

Table 1
Results from (3.9) using approximations to both Problems 4.2 and 4.3

ε	$\|u - U_h\|$	Computed error bound	γ
1	5.47e−3	6.00e−3[a]	1.10[a]
		8.02e−3[b]	1.47[b]
10^{-1}	8.00e−2	5.08e−1	6.35
		5.43e−1	6.79
10^{-2}	4.50e−1	1.21	2.69
		1.42	3.16
10^{-3}	5.47e−1	1.21	2.21
		1.43	2.61
10^{-4}	5.54e−1	1.21	2.19
		1.42	2.57
10^{-5}	5.55e−1	1.21	2.18
		1.42	2.57
10^{-6}	5.55e−1	1.21	2.18
		1.42	2.57

[a] From approximations to Problem 4.3.
[b] From approximations to Problem 4.2.

Table 2
Results from (3.9) using approximations to Problem 4.3

ε	$\|u - U_h\|$	Computed error bound	γ
1	5.47e−3	6.00e−3	1.10
10^{-1}	8.00e−2	8.79e−1	10.99
10^{-2}	4.50e−1	14.54	32.30
10^{-3}	5.47e−1	46.49	85.06
10^{-4}	5.54e−1	147.03	256.27
10^{-5}	5.55e−1	464.95	837.55
10^{-6}	5.55e−1	1470.30	2648.16

variational principle, we showed that the error in a norm equivalent to the natural energy norm is bounded by the sum of the weighted L^2-norms of solutions to a set of independent variational problems, each defined on only one element. These local problems can be solved numerically by a finite element method on a refined mesh of the original. It was shown that the numerical solutions of the local variational problems define an approximate upper error bound which differs from the theoretical upper error bound by a higher order term than the true error. Numerical results, presented to validate the method, showed that the computed error bounds are tight and always over-estimate the true errors.

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